Calcus II: For Physical Sciences and Engineering (AS.110.109)

Lecture: 1

Lecturer: Hunter Stufflebeam

Textbook Sections: 5.1, 5.2, 5.3

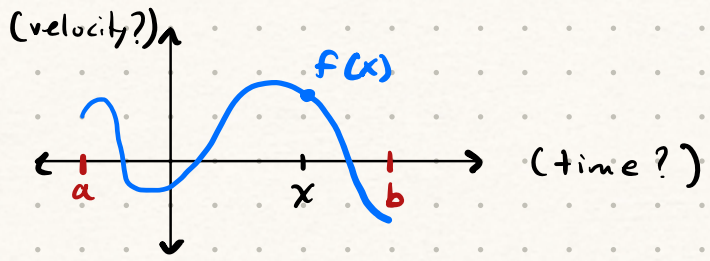
Preliminaries: For today, we will briefly review the following pre-recs:

- **Section 5.1:** Riemann sums (examples: approximating area under a curve, distance traveled in a car, ...)
- **Section 5.2:** Defining the definite integral using Riemann sums, and basic rules for computing integrals
- **Section 5.3:** The Fundamental Theorems of Calculus

⚠ If this refresher doesn't result in a feeling of refreshment — see the videos on canvas, read the book sections, and reach out to the course staff! ⚠

Motivating Problem:

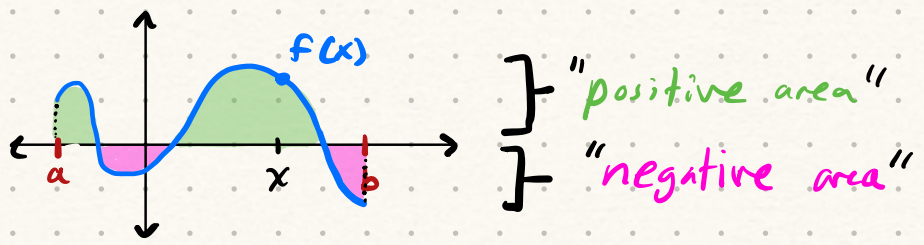
Setup: Let $[a,b]$ be an interval, and $f(x)$ a continuous function on $[a,b]$:



Ex: For example, $f(x)$ could represent the (signed) velocity of a car at time x along its journey.

In this case, x would have units of time (like seconds) and $f(x)$ would have units of velocity (like metrs/second, say moving forward if $f(x)>0$ and backwards if $f(x)<0$).

Question: What is the (signed) area trapped by the graph of $f(x)$ on the interval $[a,b]$?

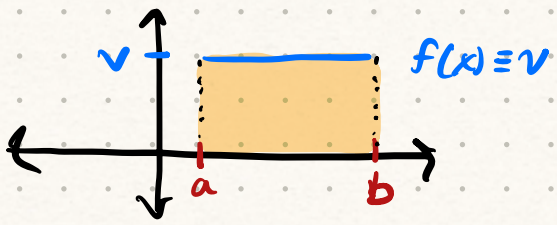


$$\text{signed area} = (\text{positive area}) - (\text{negative area})$$

Ex: In our car example, this will correspond to the displacement of the car from its original position after traveling from time a to time b.

The key idea is to remember that if the car moves at a constant velocity $f(x) \equiv v$, then

$$\text{displacement} = v \cdot (\text{travel time}) = v \cdot (b-a)$$

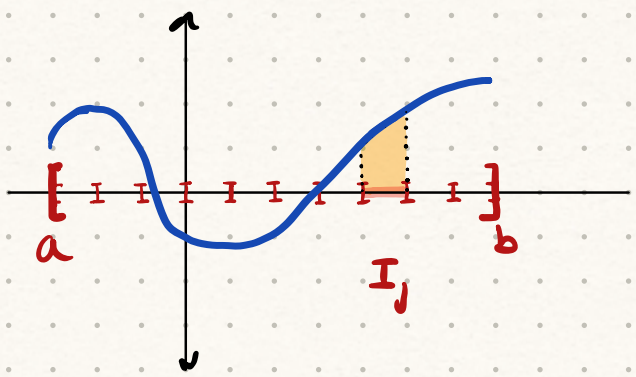


If the car's velocity, $f(x)$, changes in time, then we need to find a way to compute the signed area under the graph of $f(x)$ to compute the car's total displacement.

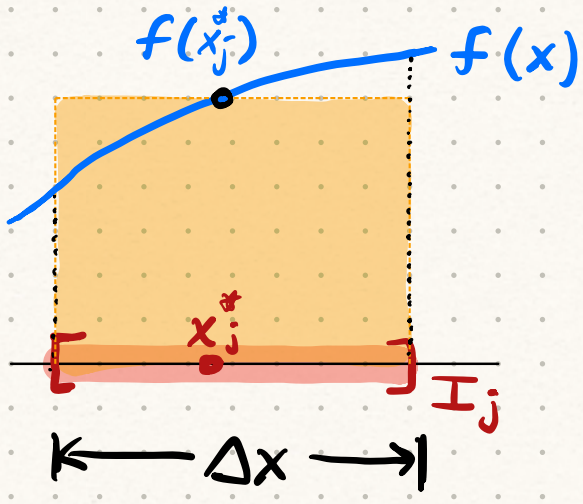
To get a good idea of how to proceed, imagine that we split the travel time interval $[a,b]$ into many (say some huge integer $N \in \{1,2,3,\dots\}$) short intervals $I_1, I_2, \dots, I_N$, each of length

$$\Delta x = \frac{b-a}{N}.$$

In each little interval I_j, pick a time x_j^* and measure the car's velocity $f(x_j^*)$



in detail



Since I_j is a very short time interval, the velocity $f(x) \approx f(x_j^*)$ for any time x in I_j, so that

$$(\text{true displacement on } I_j) \approx f(x_j^*) \cdot \Delta x = f(x_j^*) \cdot \frac{b-a}{N}$$

key observations

$$= \begin{pmatrix} \text{area of rectangle with base } I_j \\ \text{and height } f(x_j^*) \end{pmatrix}$$

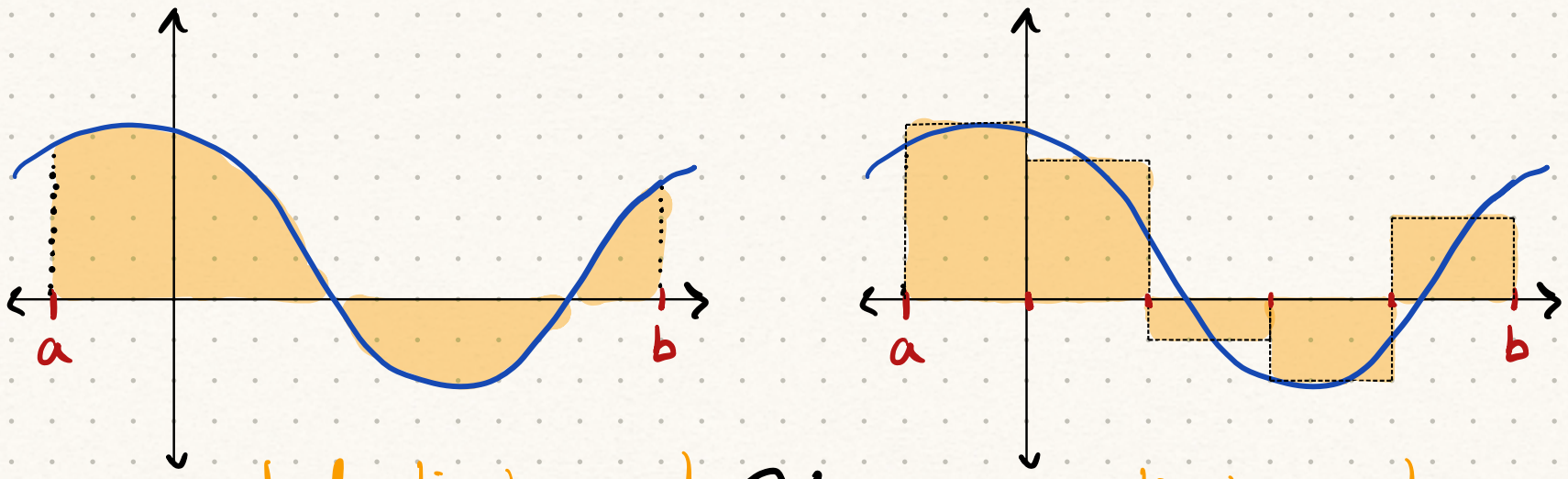
$$\approx (\text{signed area under } f \text{ on } I_j)$$

By doing this for each $I_1, I_2, \dots, I_N$ and adding the results, we get an approximation to the car's total displacement after traveling for the entire time interval $[a,b]$:

$$(\text{total displacement}) \approx \underbrace{f(x_1^*) \cdot \Delta x}_{\text{contribution from interval } I_1} + \underbrace{f(x_2^*) \cdot \Delta x}_{\text{etc}} + \dots + \underbrace{f(x_N^*) \cdot \Delta x}_{\dots \text{etc}}$$

$$= \sum_{j=1}^{N} f(x_j^*) \cdot \Delta x$$

$$\approx (\text{total signed area under } f \text{ on } [a,b])$$



actual displacement $\approx$ approx displacement

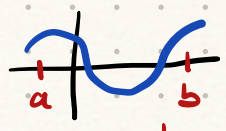
(the whole point: this is computable!)

Rmk Archimedes used similar ideas about approximation by polygons to compute the area of a circle in terms of its circumference in 250 BCE.

As we let $N \to \infty$, our approximations get better and better, and we can take this **limit** to obtain a very nice way to answer our original **Question:**

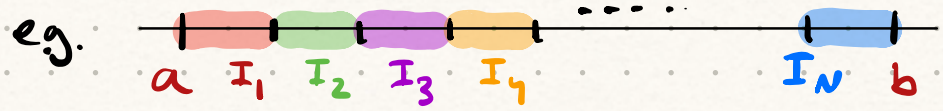
Definition / Theorem: Computing the signed area under a graph:

Let $f(x)$ be a function on the interval $[a,b]$.



① Sub-divide $[a,b]$ into N subintervals of length $\Delta x = \frac{b-a}{N}$

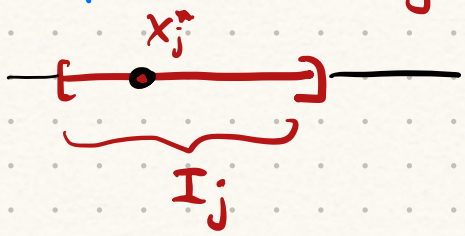
$$[a,b] = I_1 \cup I_2 \cup \cdots \cup I_N$$



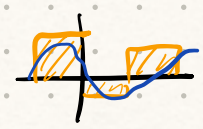
⚠ We often write each $I_j = [x_j, x_{j+1}]$ explicity in terms of its endpoints x_j.

⚠ Sometimes we even let the I_j have different lengths as long as they are all small — it doesn't end up making a difference!

② Pick a "sample point" x_j^* somewhere inside each I_j



③ Compute the **Riemann sum** $\sum_{j=1}^{N} f(x_j^*) \cdot \Delta x$



④ Take the **limit** $N \to \infty$ of these sums to obtain a number called the **definite integral** of $f(x)$ over $[a,b]$:

$$\int_a^b f(x)\,dx = \lim_{N \to \infty} \sum_{j=1}^{N} f(x_j^*)\,\Delta x.$$

$$= \text{signed area under } f \text{ on } [a,b].$$

Fact:

- This limit might NOT EXIST!
- If it DOES, then we say f is integrable on $[a,b]$.
- If $f(x)$ is continuous on $[a,b]$, or only has finitely many jump discontinuities, then f is integrable on $[a,b]$!

⚠ In recitation, we'll go through several hands-on examples, so you can refresh on the arithetical mechanics.

1) Approximate the area under $F(x)=x^2$ on $[0,1]$ using Riemann sums with $N=2, 3$ and some choice of sample points. Then think about the limit as $N\to\infty$.

2) Compute and interpret the limit

$$\lim_{N\to\infty} \sum_{i=1}^{N} \left(\frac{i}{N^2} - \frac{i^2}{N^3}\right) \quad \left(= \int_0^1 (x-x^2)\,dx\right)$$

3) Compute $\int_{-1}^{5} (1+3x)\,dx$ using Riemann sums

Properties of the definite integral

Recall the following fundamental properties, which help us efficiently compute lots of integrals:

1) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
2) $\int_a^a f(x)\,dx = 0$
3) $\int_a^b c\,dx = c(b-a)$ (for any constant function c)
4) $\int_a^b \{f(x) \pm g(x)\}\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5) $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

6) $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$ (if $a \le b \le c$)

7) If $f(x) \le g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

⚠ We'll do problems in recitation and homework 1 to get used to using these properties!

The Fundamental Theorems of Calculus:

Lastly for our review, recall the following theorems, which tell us

1) how integrals relate to derivatives.
2) how to efficiently compute definite integrals

FTC I: (Barrow, Newton, Leibniz) Let $f(x)$ be a continuous function on $[a,b]$, and for $x \in [a,b]$, define the function

$$F(x) = \int_a^x f(t)\,dt = \text{definite integral of } f \text{ from } a \text{ to the variable } x$$

⚠ Note the use of t as a dummy variable of integration, so as to not overload x as the free/independent variable.

Then $F(x)$ is a continuous function of x on $[a,b]$, and is a differentiable function of x on (a,b) with $F'(x) = f(x)$.

Thus, differentiation "undoes" integration.

To see the other direction, we have:

FTC II: (Barrow, Newton, Leibniz) Let $f(x)$ be a function on $[a,b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any anti-derivative of $f(x)$ on $[a,b]$ — i.e. any cts. function $F(x)$ on $[a,b]$ such that $F'(x) = f(x)$ on (a,b)